

The branch processes of vortex filaments and Hopf Invariant Constraint on Scroll Wave

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In this paper, by making use of Duan's topological current theory, the evolution of the vortex filaments in excitable media is discussed in detail. The vortex filaments are found generating or annihilating at the limit points and encountering, splitting, or merging at the bifurcation points of a complex function $Z(\vec{x}, t)$. It is also shown that the Hopf invariant of knotted scroll wave filaments is preserved in the branch processes (splitting, merging, or encountering) during the evolution of these knotted scroll wave filaments. Furthermore, it also revealed that the "exclusion principle" in some chemical media is just the special case of the Hopf invariant constraint, and during the branch processes the "exclusion principle" is also protected by topology.

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I. INTRODUCTION AND MOTIVATION

Scroll waves are three-dimensional (3D) extensions of the familiar spiral waves of excitable media. They have been observed in a variety of physical, chemical and biological systems[1, 2, 3, 4, 5, 6]. Recently, scroll wave have drawn great interest due to its importance in the mechanism of some re-entrant cardiac arrhythmias and fibrillation which is the leading cause of death in the industrialized world[7, 8, 9]. The scroll wave rotate about a linelike filaments called vortex filament, and usually can be defined in terms of a phase singularity. In three-dimensional excitable media, the vortex filament is commonly a closed ring, and these vortex filaments can form linked and knotted rings which contract to compact, particle-like bundles[10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

Control of scroll wave is a more important and complex problem for all excitable media. The dynamics of a 3D scroll wave are determined not only by properties of the excitable media but also by the geometry and topology of vortex filament[6, 7, 8, 9]. This implies that the control of scroll wave should stronger depend on our understanding on scroll wave topology. This inspirits us to use the topological viewpoint to study the scroll wave topology. In previous works[15, 16, 17, 18, 19, 20, 21, 22, 23], many authors have made great contributions to this issue and employed topological arguments to understand scroll wave. Some most important topological constraints on behaviors of the vortex filaments have been investigated. These topological rules may have some important applications in practice. In particular, the topological constraint on knotted vortex filaments is believed to relate to topological characteristic numbers of knotted vortex filament family, such as the winding, the self-linking and

the linking numbers. In Ref.[19], Winfree and Strogatz have proposed an "exclusion principle" which governed the scroll wave knotting and linking through each other in chemical system. This exclusion principle gives the constraint on the linking numbers and winding numbers of scroll wave.

Recently, Duan's topological current theory[15, 16, 23, 24, 25, 26] has been applied to study the topological properties of spiral waves and scroll waves. Zhang et al.[15] presented a rigorous topological description of spiral waves and scroll waves. They derived precise expressions of spiral wave and scroll wave topological charge density. Based on their work, we study the branch process of the spiral waves and calculate the knotted invariant for knotted vortex filaments by using the Duan's topological current theory, we proposed that the knotted invariant (which is just the Hopf invariant) may imply a new topological constraint on scroll wave[16, 23]. However, in Ref.[15] and in our previous work[16], the discussions are based on an important regular condition $D(\frac{\phi}{x}) \neq 0$. When this condition fails, what will happen? Main purpose of this paper is to detail this problem.

In this paper, by making using Duan's topological theory, firstly, we will extend our branch theory of spiral wave in 2D to 3D scroll wave, and study the generating, annihilating, colliding, splitting and merging of vortex filaments from a topology viewpoint. Secondly, based on the branch process of vortex filaments, it is showed that the Hopf invariant of knotted scroll wave is preserved in the branch process. This is consistent with our proposal that the Hopf invariant may implies a new constraint on scroll wave. Third, it is also shown that the "exclusion principle" in some chemical media is just the special case of the Hopf invariant constraint, and during the branch processes the "exclusion principle" is also protected by topology.

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II. TOPOLOGICAL STRUCTURES OF VORTEX FILAMENTS

In order to maintain the continuity of the whole work and make the background of this paper clear, in this section, we give a brief review of the topological current theory of vortex filaments. We chose to work with a general two-variable reaction-diffusion system whose mathematical description in terms of a nonlinear partial differential equation. This equation is written as

$$\begin{aligned}\partial_t u &= f(u, v) + D_u \nabla^2 u, \\ \partial_t v &= g(u, v) + D_v \nabla^2 v,\end{aligned}\quad (1)$$

where u and v represent the concentrations of the reagents; ∇^2 is the Laplacian operator in three-dimensional space; $f(u, v)$ and $g(u, v)$ are the reaction functions. Following the description in Ref.[15, 27], we define a complex function $Z = \phi^1 + i\phi^2$, where $\phi^1 = u - u^*$ and $\phi^2 = v - v^*$. Here u^* and v^* are the concentrations of the vortex filaments.

We know that the complex function $Z = \phi^1 + i\phi^2$ can be regarded as the complex representation of a two-dimensional vector field $\vec{Z} = (\phi^1, \phi^2)$. Let us define the unit vector: $n^a = \frac{\phi^a}{\|\phi\|}$ ($a = 1, 2$; $\|\phi\|^2 = \phi^a \phi^a = Z^* Z$). It is easy to see that the zeros of Z are just the singularities of \vec{n} . Using this unit vector \vec{n} , an “induced abelian gauge potential” can be constructed with

$$\begin{aligned}A_\mu &= \epsilon_{ab} n^a \partial_\mu n^b, \quad \mu = 0, 1, 2, 3; \\ \partial_\mu &= (\partial_0, \nabla), \quad \partial_0 = \partial_t = \frac{\partial}{\partial t},\end{aligned}\quad (2)$$

the gauge field strength given by this gauge potential is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (3)$$

According to Ref.[15, 16, 23, 24, 25, 26], the two dimensional topological tensor current is defined as

$$K^{\mu\nu} = \frac{1}{4\pi} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda\rho} \partial_\lambda n^a \partial_\rho n^b. \quad (4)$$

It is easy to show that the topological tensor current $K^{\mu\nu}$ can be rewritten in a compact form

$$K^{\mu\nu} = \delta^2(\vec{\phi}) D^{\mu\nu} \left(\frac{\phi}{x} \right), \quad (5)$$

where $D^{\mu\nu}$ is the general Jacobian determinants

$$\epsilon^{ab} D^{\mu\nu} \left(\frac{\phi}{x} \right) = \epsilon^{\mu\nu\lambda\rho} \partial_\lambda \phi^a \partial_\rho \phi^b. \quad (6)$$

Defining the spatial components of $K^{\mu\nu}$ as

$$j^i = K^{0i} = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{ab} \partial_j n^a \partial_k n^b, \quad i, j, k = 1, 2, 3, \quad (7)$$

we have

$$j^i = \delta(\vec{\phi}) D^i \left(\frac{\phi}{x} \right), \quad (8)$$

where $D^i \left(\frac{\phi}{x} \right) = \frac{1}{2} \epsilon^{ijk} \epsilon_{ab} \partial_j \phi^a \partial_k \phi^b$ is the Jacobian vector. This delta function expression of the topological current j^i tells us it doesn't vanish only when the vortex filaments exist, i.e.,

$$j^i \begin{cases} = 0, & \text{if and only if } \vec{\phi} \neq 0; \\ \neq 0, & \text{if and only if } \vec{\phi} = 0. \end{cases} \quad (9)$$

So the sites of the vortex filaments determine the nonzero solutions of j^i . The implicit function theory shows that under the regular condition[28]

$$D^{\mu\nu} \left(\frac{\phi}{x} \right) \neq 0, \quad (10)$$

the general solutions of

$$\phi^1(t, \vec{x}) = 0, \quad \phi^2(t, \vec{x}) = 0 \quad (11)$$

can be expressed as

$$x^1 = x_l^1(t, s), \quad x^2 = x_l^2(t, s), \quad x^3 = x_l^3(t, s), \quad (12)$$

which represent the world surface of N moving isolated vortex filaments with string parameter s ($l = 1, 2, \dots, N$). These singular strings solutions are just the vortex filaments. In delta function theory[29], one can prove that in three-dimensional space,

$$\delta(\vec{\phi}) = \sum_{l=1}^N \beta_l \int_{L_l} \frac{\delta^3(\vec{x} - \vec{x}_l(s))}{|D(\frac{\phi}{u})|_{\Sigma_l}} ds, \quad (13)$$

where $D(\frac{\phi}{u}) = \frac{1}{2} \epsilon^{jk} \epsilon_{mn} \frac{\partial \phi^m}{\partial u^j} \frac{\partial \phi^n}{\partial u^k}$ and Σ_l is the l -th planar element transverse to L_l with local coordinates (u^1, u^2) . The positive integer β_l is the Hopf index of ϕ -mapping, which means that when \vec{x} covers the neighborhood of the zero point $\vec{x}_l(s, t)$ once, the vector field $\vec{\phi}$ covers the corresponding region in ϕ space for β_l times. Meanwhile the direction vector of L_l is given by[25, 26]

$$\frac{dx^i}{ds} \Big|_{\vec{x}_l} = \frac{D^i(\phi/x)}{D(\phi/u)} \Big|_{\vec{x}_l}. \quad (14)$$

Then considering Eqs.(13) and Eqs.(14), we obtain the inner structure of j^i ,

$$\begin{aligned}j^i &= \delta(\vec{\phi}) D^i \left(\frac{\phi}{x} \right) \\ &= \sum_{l=1}^N \beta_l \eta_l \int_{L_l} dx^i \delta^3(\vec{x} - \vec{x}_l),\end{aligned} \quad (15)$$

where $\eta_l = \text{sgn} D(\frac{\phi}{u}) = \pm 1$ is the Brouwer degree of ϕ -mapping, with $\eta_l = 1$ corresponding to the vortex filament and $\eta_l = -1$ corresponding to the antivortex filament. We find that the topological current \vec{j} is just the charge density vector $\vec{\rho}$ of the vortex filament in Ref.[15]. In our theory, the topological charge of the vortex filament L_l is

$$Q_l = \int_{\Sigma_l} \vec{j} \cdot d\vec{\sigma} = W_l = \beta_l \eta_l, \quad (16)$$

in which W_l is just the winding number of $\vec{\phi}$ around L_l , the above expression reveals distinctly that the topological charge of vortex filament is not only the winding number, but also expressed by the Hopf indices and Brouwer degrees. The topological inner structure showed in Eq.(16) is more essential than usually considered and it will be helpful as a complement of the current description of scroll wave only by winding number in topology. This shows that the advantage of our topological description of the vortex filaments.

In the above, we give an prime introduction of the topological current theory of scroll wave, which is based on the rigorous mathematics background that we called Duan's topological current theory. Here we give some remarks about above results in order. (i). The definition of the complex function $Z = \phi^1 + i\phi^2$ originates from u and v . Therefore, Z is the function that shows the concentration distribution of the reagents. The exact expression of Z is determined by the nonlinear partial differential equation (1). If the exact analytical solution of Eq.(1) is known, we can directly calculate the zero points of Z and determine the Hopf indices and Brouwer degrees, which describe the inner topological structures of the vortex filaments of scroll wave. (ii). The traditional theoretical description of scroll wave usually used the phase singularity method and cannot directly deal with the zero points of Z . The above results show that the topological current theory of scroll wave provide a available approach to deal with the zero points. (iii). The regular condition (10) will plays a essential role in determining the stability of the scroll wave. This condition also depends on the exact solution of the Eq.(1). It can be determined when the expression of Z is known. How this condition determine the stability of the scroll wave is the tasks of the following sections.

III. THE BRANCH PROCESS OF VORTEX FILAMENTS AT THE LIMITED POINT

However, from the above discussion we know that the results mentioned are obtained under the condition $D^{\mu\nu}(\frac{\phi}{x}) \neq 0$. When this condition fails, i.e., the Brouwer degrees η_l are indefinite, what will happen? In what follows, we will study the case when $D^{\mu\nu}(\frac{\phi}{x}) = 0$. It often happens when the zero of \vec{Z} includes some branch points, which lead to the bifurcation of the topological current.

Generally speaking, the evolution of a vortex filament L_l can be discussed from Eq.(5). From Eq.(4), considering that $\epsilon^{\mu\nu\lambda\rho}$ is a fully antisymmetric tensor, we can prove that

$$\partial_\mu K^{\mu\nu} = 0, \quad (17)$$

that is

$$\partial_0 j^i + \partial_j K^{ji} = 0. \quad (18)$$

This is the continuity equation constraints on vortex filaments. In order to discuss the evolution of these vortex

filaments and simplify our study, we fixed the $x^3 = z$ coordinate and take the XOY plane as the cross section, so the intersection line between the L_l 's evolution surface and the cross section is just the motion curve of L_l . In this case the 2D topological current is defined as

$$j^3 = K^{03} = \delta^2(\phi) D^0\left(\frac{\phi}{x}\right) \quad (19)$$

and

$$K^i = K^{i3} = \delta^2(\phi) D^i\left(\frac{\phi}{x}\right), \quad i = 1, 2. \quad (20)$$

It is easy to see that j^3 and K^i satisfy the continuity equation

$$\partial_0 j^3 + \partial_i K^i = 0. \quad (21)$$

The velocity of the intersection point of vortex filament and the cross section is given by

$$\frac{dx^i}{dt} = \frac{D^i(\phi/x)}{D^0(\phi/x)}. \quad (22)$$

From Eq.(22) it is obvious that when

$$D^0(\phi/x) = 0$$

at the very point (t^*, \vec{x}^*) , the velocity

$$\frac{dx^1}{dt} = \frac{D^1(\phi/x)}{D^0(\phi/x)} \Big|_{(t^*, \vec{x}^*)}, \quad \frac{dx^2}{dt} = \frac{D^2(\phi/x)}{D^0(\phi/x)} \Big|_{(t^*, \vec{x}^*)} \quad (23)$$

is not uniquely determined in the neighborhood of (t^*, \vec{x}^*) . This critical point is called the branch point. In Duan's topological current theory usually there are two kinds of branch points, namely the limit points and the bifurcation points, each kind of which corresponds to different cases of branch process.

First, in this section, we only study the case that the zeros of the complex function \vec{Z} includes some limit points which satisfy

$$D^1\left(\frac{\phi}{x}\right) \Big|_{(t^*, \vec{x}^*)} \neq 0, \quad D^2\left(\frac{\phi}{x}\right) \Big|_{(t^*, \vec{x}^*)} \neq 0. \quad (24)$$

For simplicity, we assume that $D^2(\phi/x)|_{(t^*, \vec{x}^*)} \neq 0$ is always satisfied in our next discussions. When $D^1(\phi/x)|_{(t^*, \vec{x}^*)} \neq 0$, from Eq.(22) we obtain

$$\frac{dx^1}{dt} = \frac{D^1(\phi/x)}{D^0(\phi/x)} \Big|_{(t^*, \vec{x}^*)} = \infty; \quad (25)$$

i.e.,

$$\frac{dt}{dx^1} \Big|_{(t^*, \vec{x}^*)} = 0. \quad (26)$$

Then, the Taylor expansion of $t = t(x^1, t)$ at the limit point (t^*, \vec{x}^*) of vortex filament, one can obtain

$$t - t^* = \frac{1}{2} \frac{d^2 t}{(dx^1)^2} \Big|_{(t^*, \vec{x}^*)} (x^1 - x^{1*})^2, \quad (27)$$

which is a parabola in $x^1 - t$ plane. From Eq.(27) we can obtain two solutions $x_1^1(t)$ and $x_2^1(t)$, which give two branch solutions of vortex filament at the limit points. If

$$\left. \frac{d^2 t}{(dx^1)^2} \right|_{(t^*, \vec{x}^*)} > 0, \quad (28)$$

we have the branch solutions for $t > t^*$ [see Fig. 1(a)]; otherwise, we have the branch solutions for $t < t^*$ [see Fig. 1(b)]. The former is related to the origin of the vortex filament at the limit points, and the later is the annihilation of the vortex filament. At the neighborhood of the limit point, we denote the length scale $l = \Delta x$. From Eq.(27), one can obtain the approximation relation

$$l \propto \|t - t^*\|^{1/2}. \quad (29)$$

The growth rate $\gamma = l/\Delta x$ or annihilation rate of vortex lines is

$$\gamma \propto (t - t^*)^{-1/2}. \quad (30)$$

From the continuity equation Eq.(21), we know that the topological number of the vortex filament is identically conserved. This means that the total topological number of the final vortex filaments equals that of the initial vortex filaments. The total numbers of these two generated or annihilated vortex filaments must be zero at the limit point; i.e., the two generated or annihilated vortex filaments have to be opposite,

$$\beta_1 \eta_1 + \beta_2 \eta_2 = 0, \quad (31)$$

which shows that $\beta_1 = \beta_2$ and $\eta_1 = -\eta_2$. One can see the fact that the Brouwer degree η is indefinite at the limit points implies that it can change discontinuously at limit points.

For a limit point it is required that $D^1(\phi/x)|_{(t^*, \vec{x}^*)} \neq 0$. As to a bifurcation point[33], it must satisfy a more complex condition. This case will be discussed in the following section.

IV. THE BRANCH PROCESS OF VORTEX FILAMENTS AT THE BIFURCATION POINT

Now let us study the bifurcation of the vortex line at its bifurcation point where

$$D^0\left(\frac{\phi}{x}\right)\Big|_{(t^*, \vec{x}^*)} = 0, \quad D^1\left(\frac{\phi}{x}\right)\Big|_{(t^*, \vec{x}^*)} = 0. \quad (32)$$

These two restrictive conditions will lead to an important fact that the function relationship between t and x^1 is not unique in the neighborhood of the bifurcation point (t^*, \vec{x}^*) . The equation

$$\frac{dx^1}{dt} = \frac{D^1(\phi/x)}{D^0(\phi/x)}\Big|_{(t^*, \vec{x}^*)}, \quad (33)$$

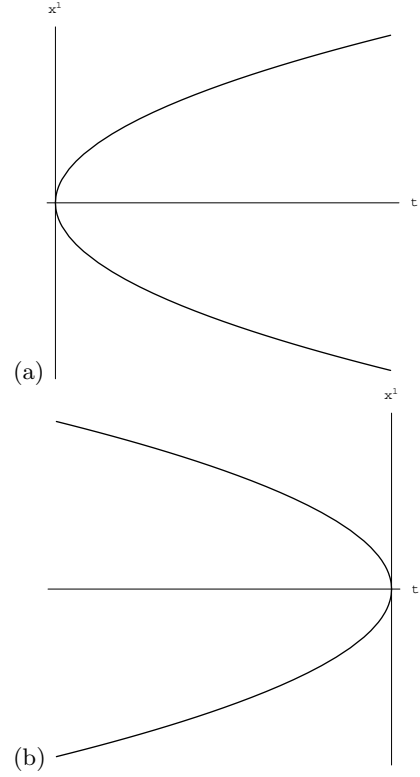


FIG. 1: We fixed the limit point (x^1, t^*) at the origin of $(x^1 - t)$ plane. (a) The branch solutions for Eq.(27) when $d^2 t/(dx^1)^2|_{(t^*, \vec{x}^*)} > 0$, i.e., a pair of vortex filaments with opposite charges generate at the limit point, i.e., the origin of vortex filaments. (b) The branch solutions for Eq.(27) when $d^2 t/(dx^1)^2|_{(t^*, \vec{x}^*)} < 0$, i.e., a pair of vortex filaments with opposite charges annihilate at the limit point.

which, under restraint of Eq.(32), directly shows that the direction of the integral curve of Eq. (33) is indefinite at the point (t^*, \vec{x}^*) . This is why the very point (t^*, \vec{x}^*) is called a bifurcation point.

Assume that the bifurcation point (t^*, \vec{x}^*) has been found from Eqs.(11) and (32). We know that, at the bifurcation point (t^*, \vec{x}^*) , the rank of the Jacobian matrix $[\partial\phi/\partial x]$ is 1. In addition, according to the Duan's topological current theory, the Taylor expansion of the solution of Eq.(11) in the neighborhood of the bifurcation point (t^*, \vec{x}^*) can be expressed as[23, 25]

$$A(x^1 - x^{*1})^2 + 2B(x^1 - x^{*1})(t - t^*) + C(t - t^*)^2 = 0 \quad (34)$$

which leads to

$$A\left(\frac{dx^1}{dt}\right)^2 + 2B\left(\frac{dx^1}{dt}\right) + C = 0 \quad (35)$$

and

$$C\left(\frac{dt}{dx^1}\right)^2 + 2B\left(\frac{dt}{dx^1}\right) + A = 0, \quad (36)$$

where A , B , and C are three constants. The solutions of Eq.(35) or Eq.(36) give different directions of the branch

curves at the bifurcation point. There are four possible cases, which will show the physical meanings of the bifurcation points.

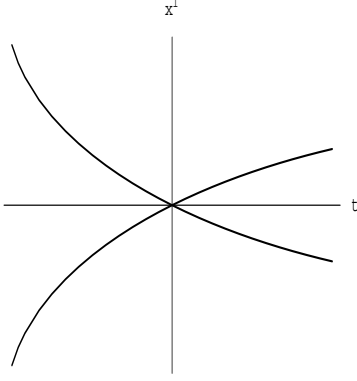


FIG. 2: We fixed the bifurcation point (x^{1*}, t^*) at the origin of $(x^1 - t)$ plane. Two vortex filaments meet and then depart at the bifurcation point.

Case 1 ($A \neq 0$). For $\Delta = 4(B^2 - AC) > 0$ from Eq.(35) we get two different directions of the velocity field of vortex filaments

$$\left. \frac{dx^1}{dt} \right|_{(t^*, \vec{x}^*)} = \frac{-B \pm \sqrt{B^2 - AC}}{A}, \quad (37)$$

Which is shown in Fig.2. It is the intersection of two vortex filaments with different directions at the bifurcation point, which means that two vortex filaments meet and then depart from each other at the bifurcation point.

Case 2 ($A \neq 0$). For $\Delta = 4(B^2 - AC) = 0$ from Eq.(35) we obtain only one direction of the velocity of vortex filaments

$$\left. \frac{dx^1}{dt} \right|_{(t^*, \vec{x}^*)} = \frac{-B}{A} \quad (38)$$

which includes three important situations. (a) Two vortex filaments tangentially encounter at the bifurcation point [See Fig.3(a)]. (b) Two vortex filaments merge into one vortex filament at the bifurcation point [See Fig.3(b)]. (c) One vortex filament splits into two vortex filaments at the bifurcation point [See Fig.3(c)].

Case 3 ($A = 0, C \neq 0$). For $\Delta = 4(B^2 - AC) = 0$, we have

$$\left. \frac{dt}{dx^1} \right|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{C} = \begin{cases} 0, \\ -\frac{2B}{C}. \end{cases} \quad (39)$$

There are two important cases: (a) One vortex filament splits into three vortex filaments at the bifurcation point [See Fig.4(a)]. (b) Three vortex filaments merge into one vortex filament at the bifurcation point [See Fig.4(b)].

Case 4 ($A = C = 0$). Equation(35) and Eq(36) give respectively

$$\frac{dx^1}{dt} = 0, \quad \frac{dt}{dx^1} = 0. \quad (40)$$

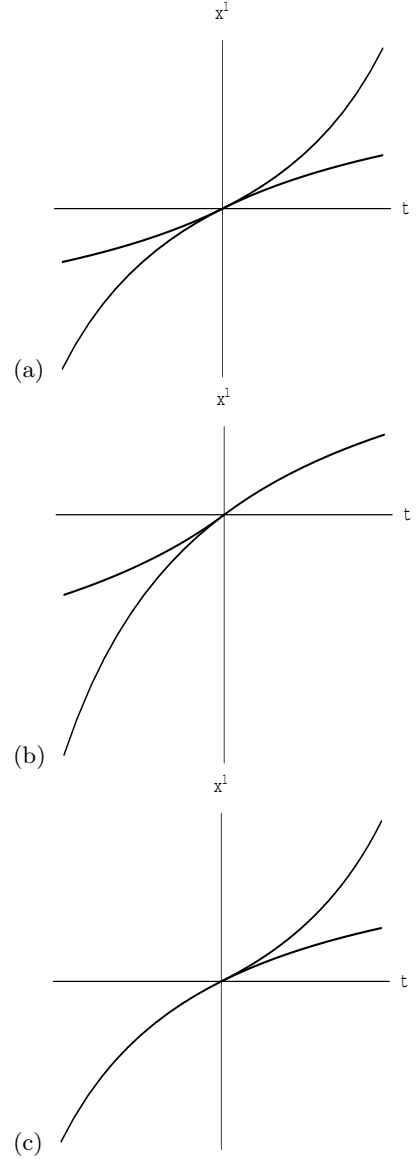


FIG. 3: We fixed the bifurcation point (x^{1*}, t^*) at the origin of $(x^1 - t)$ plane. (a) Two vortex filaments tangentially encounter at the bifurcation point. (b) Two vortex filaments merge into one vortex filament at the bifurcation point. (c) One vortex filament splits into two vortex filaments at the bifurcation point.

This case is obvious similar to Case 3, see Fig.5.

The above solutions reveal the evolution of the vortex filaments. Besides the encountering of the vortex filaments, i.e., a vortex filament pair encounter and then depart at the bifurcation point along different branch curves [See Fig.2 and Fig.3(a)], it also includes splitting and merging of vortex filaments. When a multi-charged vortex filament moves through the bifurcation point, it may split into several vortex filaments along different branch curves [See Fig.3(c), Fig.4(a) and Fig.5(b)]. On the contrary, several vortex filaments can merge into a

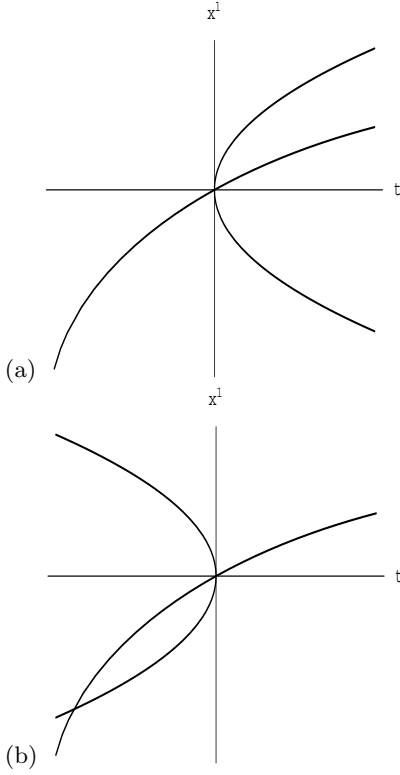


FIG. 4: We fixed the bifurcation point (x^{1*}, t^*) at the origin of $(x^1 - t)$ plane. (a) One vortex filament splits into three vortex filaments at the bifurcation point. (b) Three vortex filaments merge into one vortex filament at the bifurcation point.

vortex filament at the bifurcation point [See Fig.3(b) and Fig.4(b)].

At the neighborhood of the bifurcation point, we denote scale length $l = \Delta x$. From Eqs.(37)-(39) we can then obtain the approximation asymptotic relation

$$l \propto (t - t^*). \quad (41)$$

The growth rate γ or annihilation rate of vortex filament γ of the vortex filament is

$$\gamma \propto \text{const.} \quad (42)$$

From Eq.(40), one can obtain

$$l = \text{const}, \quad \gamma = 0. \quad (43)$$

It is obvious that the vortex filaments are relatively at rest when $l = \text{const}$.

The identical conversation of the topological charge shows the sum of the topological charge of these final vortex filaments must be equal to that of the original vortex filaments at the bifurcation point, i.e.,

$$\sum_i \beta_{l_i} \eta_i = \sum_f \beta_{l_f} \eta_f \quad (44)$$

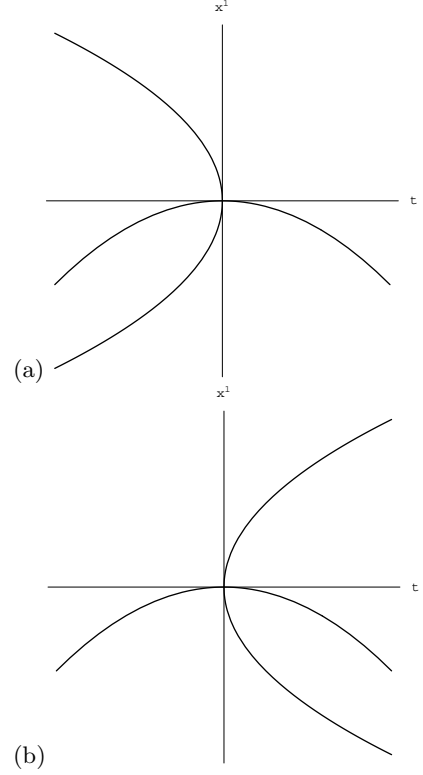


FIG. 5: We fixed the bifurcation point (x^{1*}, t^*) at the origin of $(x^1 - t)$ plane. In this case, Two vortex filaments intersect normally at the bifurcation point. (a) Three vortex filaments merge into one vortex filament at the bifurcation point. (b) One vortex filament splits into three vortex filaments at the bifurcation point.

for fixed l . Furthermore, from the above studies, we see that the generation, annihilation, and bifurcation of vortex filaments are not gradually changed, but suddenly changed at the critical points.

V. HOPF INVARIANT CONSTRAINT ON SCROLL WAVE

In this section, we will research the topological properties of the knotted vortex filaments. We first consider the continuity equation constraint, from Eq.(7) one can obtain

$$\partial_i j^i = 0, \quad (45)$$

which can also be derived from Eq.(17). The continuity equation (45) implies that the vortex filament may be either closed loops or infinite curves. In Ref.[15], zhang et.al. pointed that the continuity equation (45) is consistent with the topological rule which governs the scroll wave pinning to an inclusion[17].

Except the continuity equation constraint on scroll wave, the complex scroll wave topology may provide

other topological requirements on scroll wave. In the following discussions in this section, we will study an important knotted invariant which constraints on scroll wave. It is well known that the Hopf invariant is an important topological invariant to describe the topological characteristics of the knot family. In our topological theory of knotted vortex filaments, the Hopf invariant relates to the topological characteristic numbers of the knotted vortex filaments family. In a closed three-manifold M the Hopf invariant is defined as [16, 24]

$$H = \frac{1}{2\pi} \int_M A \wedge F = \frac{1}{2\pi} \int_M A_i j^i d^3x. \quad (46)$$

Substituting Eq.(15) into Eq.(46), one can obtain

$$H = \frac{1}{2\pi} \sum_{l=1}^N W_l \int_{L_l} A_i dx^i. \quad (47)$$

It can be seen that when these N vortex filaments are N closed curves, i.e., a family of N knots $\xi_l (l = 1, 2, \dots, N)$, Eq.(47) leads to

$$H = \frac{1}{2\pi} \sum_{l=1}^N W_l \oint_{\xi_l} A_i dx^i. \quad (48)$$

This is a very important expression. Consider a transformation of complex function $Z' = e^{i\theta} Z$, this gives the $U(1)$ gauge transformation of $A_i : A'_i = A_i + \partial_i \theta$, where $\theta \in R$ is a phase factor denoting the $U(1)$ gauge transformation. It is seen that the $\partial_i \theta$ term in Eq.(48) contributes nothing to the integral H when the vortex filaments are closed, hence the expression (48) is invariant under the $U(1)$ gauge transformation. As pointed out in Ref.[15], a singular vortex filament is either closed ring or infinite curve, therefore we conclude that the Hopf invariant is a spontaneous topological invariant for the vortex filaments in excitable media.

According to our previous work in Ref.[16], a precise expression of the Hopf invariant is

$$H = \sum_{k=1}^N W_k^2 SL(\xi_k) + \sum_{k,l=1 (k \neq l)}^N W_k W_l Lk(\xi_k, \xi_l), \quad (49)$$

where $Lk(\xi_k, \xi_l)$ is the Gauss linking number between different knotted vortex filaments ξ_k and ξ_l , and $SL(\xi_k)$ is the self-linking number of closed filament ξ_k with an imaginary closed filament infinitesimally nearby [30, 31, 32]. The Eq.(49) reveals the relationship between H and the self-linking and the linking numbers of the vortex filaments knots family. Since the self-linking and the linking numbers are both the invariant characteristic numbers of the vortex filaments knots family in topology, H is an important topological invariant required to describe the linked vortex filaments in excitable media.

In the following we will discuss the conservation of the Hopf invariant in the branch processes of knotted filaments. In the branch process of vortex filament, we note

that the sum of the topological charges of final vortex filaments must be equal to that of the initial vortex filaments at the bifurcation point. This conclusion is always valid because it is in topological level. So we have,

- (a) for the case that one filament ξ split into two filaments ξ_1 and ξ_2 , we have $W_\xi = W_{\xi_1} + W_{\xi_2}$;
- (b) two vortex filaments ξ_1 and ξ_2 merge into one filament: $W_{\xi_1} + W_{\xi_2} = W_\xi$;
- (c) two vortex filaments ξ_1 and ξ_2 meet, then depart as other two filaments ξ_3 and ξ_4 : $W_{\xi_1} + W_{\xi_2} = W_{\xi_3} + W_{\xi_4}$.

In the following we will show that when the branch processes of knotted vortex filaments occur as above, the Hopf invariant is preserved:

(A) *The splitting case.* We consider one knot ξ split into two knots ξ_1 and ξ_2 which are of the same self-linking number as ξ ($SL(\xi) = SL(\xi_1) = SL(\xi_2)$). And then we will compare the two number H_ξ and $H_{\xi_1+\xi_2}$ (where H_ξ is the contribution of ξ to H before splitting, and $H_{\xi_1+\xi_2}$ is the total contribution of ξ_1 and ξ_2 to H after splitting. First, from the above text we have $W_\xi = W_{\xi_1} + W_{\xi_2}$ in the splitting process. Second, on the one hand, noticing that in the neighborhood of bifurcation point, ξ_1 and ξ_2 are infinitesimally displace from each other; on the other hand, for a knot ξ its self-linking number $SL(\xi)$ is defined as

$$SL(\xi) = Lk(\xi, \xi_V), \quad (50)$$

where ξ_V is another knot obtained by infinitesimally displacing ξ in the normal direction \vec{V} [34]. Therefore

$$SL(\xi) = SL(\xi_1) = SL(\xi_2) = Lk(\xi_1, \xi_2) = Lk(\xi_2, \xi_1) \quad (51)$$

and

$$Lk(\xi, \xi'_k) = Lk(\xi_1, \xi'_k) = Lk(\xi_2, \xi'_k) \quad (52)$$

(where ξ'_k denotes another arbitrary knot in the family ($\xi'_k \neq \xi, \xi'_k \neq \xi_{1,2}$)). Then, third, we can compare H_ξ and $H_{\xi_1+\xi_2}$ before splitting,

$$H_\xi = W_\xi^2 SL(\xi) + \sum_{k=l(\xi'_k \neq \xi)}^N 2W_\xi W_{\xi'_k} Lk(\xi, \xi'_k), \quad (53)$$

where $Lk(\xi, \xi'_k) = Lk(\xi'_k, \xi)$; after splitting,

$$\begin{aligned} H_{\xi_1+\xi_2} &= W_{\xi_1}^2 SL(\xi_1) + W_{\xi_2}^2 SL(\xi_2) + 2W_{\xi_1} W_{\xi_2} Lk(\xi_1, \xi_2) \\ &+ \sum_{k=l(\xi'_k \neq \xi_{1,2})}^N 2W_{\xi_1} W_{\xi'_k} Lk(\xi_1, \xi'_k) \\ &+ \sum_{k=l(\xi'_k \neq \xi_{1,2})}^N 2W_{\xi_2} W_{\xi'_k} Lk(\xi_2, \xi'_k). \end{aligned} \quad (54)$$

Comparing Eqs.(53) and (54), we have

$$H_\xi = H_{\xi_1+\xi_2} \quad (55)$$

This means that in the splitting process the Hopf invariant is conserved.

(B) *The mergence case.* We consider two knots ξ_1 and ξ_2 , which are of the same self-linking number, merge into one knot ξ which is of the same self-linking number as ξ_1 and ξ_2 . This is obviously the inverse process of the above splitting case, therefore we have

$$H_{\xi_1+\xi_2} = H_{\xi}. \quad (56)$$

(C) *The intersection case.* This case is related to the collision of two knots. we consider that two knots ξ_1 and ξ_2 , which are of the same self-linking number, meet, and then depart as other two knots ξ_3 and ξ_4 which are of the same self-linking number as ξ_1 and ξ_2 . This process can be identified to two sub-processes: ξ_1 and ξ_2 merge into one knot ξ , and then ξ split into ξ_3 and ξ_4 . Therefore, from the above two cases (B) and (A) we have

$$H_{\xi_1+\xi_2} = H_{\xi_3+\xi_4} \quad (57)$$

Therefore we acquire the result that, in the branch processes during the evolution of knotted vortex filaments (splitting, mergence, and intersection), the Hopf invariant is preserved.

The above analysis show that the branch processes of knotted vortex filament family must satisfy the Hopf invariant constraint. This conclusion is obtained only from the viewpoint of topology without using any particular models or hypothesis. Therefore, the Hopf invariant is a more extensive topological constraint on scroll wave, and it is valid in almost systems which support the existence of scroll wave.

According to Winfree and Strogatz[19], there is an “exclusion principle” governed the scroll wave knotting and linking through each other in chemical system. The chemical requirement plays a crucial role in such “exclusion principle”. It states that the topology of scroll wave in such system must satisfy the constraints that:

$$W_k SL(\xi_k) + \sum_{l=1(l \neq k)}^N W_l Lk(\xi_k, \xi_l) = 0. \quad (58)$$

It is very easy to see that the “exclusion principle” makes the Hopf invariant trivially, i.e., $H = 0$. This is just a special case of the Hopf invariant constraint. When branch processes of scroll wave occur, it is obvious from above discussion that the “exclusion principle” is also protected by topology.

VI. CONCLUSION AND DISCUSSIONS

First, we give a prime review of the topological theory of vortex filaments in three dimensional excitable media. When $D(\phi/x) = 0$, the intersection, splitting, and merging of line defects in three-dimensional space are investigated in detail by making using of Duan’s topological current theory. Second, the evolution of vortex filaments in $(3+1)$ -dimensional space-time is studied. There exist crucial cases of branch processes in the evolution of

vortex filaments when the Jacobian $D(\phi/x) = 0$, i.e., η_l is indefinite. At one of the limit points of the complex function Z , a pair of vortex filaments with opposite topological charge can be annihilated or generated. At one of the bifurcation points of Z , a vortex filament with topological charge W_{ξ} may split into several vortex filaments (total topological charges is W_{ξ}); conversely, several vortex filaments (total topological charges is W_{ξ}) can merge into one vortex filament with a topological charge W_{ξ} . Also, at one of the bifurcation points of Z , two filaments meet and then depart. These show that vortex filaments are unstable at these branch points of Z . From the topological properties of the complex function Z , we obtained that the velocity of the vortex filaments is infinite when they are being annihilated or generated, which agrees with the similar results of line defects what was obtained by Bray and Mazenko[35]. The velocity of the vortex filament at the limit point or bifurcation point has been shown clearly in Fig.1 to Fig.5 (The slope of the curve). Third, based on the branch process of vortex filaments, it is showed detailed that the Hopf invariant of knotted scroll wave is preserved in the branch process, this is consistent with our proposal that the Hopf invariant may implies a new constraint (the Hopf invariant constraint) on scroll wave. Furthermore, it also revealed that the “exclusion principle” in some chemical media is just the special case of the Hopf invariant constraint, and during the branch processes the “exclusion principle” is also protected by topology. Finally, we would like to point out that both branch theory of vortex filaments and the Hopf invariant constraint in this paper are obtained from only the viewpoint of topology, without using any particular models or hypotheses. These results are valid for all systems which support the existence of scroll wave.

In this paper, we give a rigorous and general topological investigation of scroll wave. This work can be applied in practice. Here we give some discussions about how to connect our work to the phenomenology and mathematical analysis of scroll wave. (i). The regular condition (10) can be regarded as stability condition of scroll wave. When the solutions of Eq.(1) is determined, the exact expression of (10) can be work out. In this case, we can directly calculate the stability condition of scroll wave. This provide a direct approach to investigate the stability of scroll wave and how to control it. (ii). When the regular condition (10) fails, the scroll wave is unstable. In this case, the branch processes occur. The branch condition $D(\phi/x) = 0$ predicts where and how these processes will occur, if the scroll wave solution of Eq.(1) is work out. In laboratory, these branch processes can be produced by using the branch condition. This provide an experimental approach to test our work. (iii) In this paper, we have predicted that the velocity of the vortex filaments is infinite when they are being annihilated or generated. The similar phenomena has been obtained in the phase-ordering system[35]. We expect that this phenomena will be observed for scroll wave in laboratory. (iv). For knotted vortex filaments, the Hopf invariant

will play an important role in control the behaviours of scroll wave. During the branch processes, the Hopf invariant will protect by topology. We also hope that this will be proved by experiment. (v). At last, we point out that Eq.(1) determines the solution of Z , and of course it determines the regular condition and branch condition. The investigation presents in this paper is based on an important precondition that the scroll wave solutions can be worked out from Eq.(1).

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